A CALCULUS WORKBOOK

The Language of Calculus

Functions, algebra, and trigonometry, practiced until you can do them without thinking. That is most of the work in any calculus problem, and it is where most points are lost.

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Why this workbook exists

Every calculus professor knows a secret they rarely say out loud: **students do not fail calculus on the calculus.** They fail on rusty algebra and half-remembered trig. In a typical problem, about 90% of the work is algebra and only one step is the genuinely new idea. Miss the algebra and you lose the mark on a line that has nothing to do with calculus.

This workbook covers that 90% before Calculus I begins. Thirty-three lessons on **functions, algebra, and trigonometry**, practiced until you can do them without thinking. Every lesson opens with a picture to give you the idea, then ends in problems you solve yourself, because you cannot learn this by watching. Every answer leads with the result and then explains it, so you check yourself at a glance and study only what you missed. The last three lessons go through the doorway into calculus itself: limits, continuity, and the derivative.

Work the problems on the ruled space, then check them against the answers that follow. Do them with no notes. A half-finished answer counts as a miss, because in calculus it is one.

Master the fundamentals, and all that is left is the calculus.

Who wrote this

I am a college student majoring in computer engineering. I built this workbook for myself after finding out where Calculus I actually goes wrong. It is not the calculus. It is the algebra and trigonometry underneath it. I designed the sequence, wrote the diagnostic, solved every problem myself, and rewrote the lessons that did not work. I am not a professor, and that is the point. I sat exactly where you are, and this is the preparation I wish someone had handed me.

Special thanks

This workbook exists because of **DIBEOS**, whose roadmap for calculus makes the case that the language of calculus is the stage most people skip. This workbook is that stage.

> **This is the free preview: 5 lessons of 33.** You get the Day-1 benchmark, the first factoring lesson, what a function really is, sine & cosine, and the first honest look at a limit. The full workbook adds the rest of the Algebra Toolkit, every parent function, all of trigonometry, rational expressions, continuity and the derivative, three mixed checkpoints, and a readiness final that rematches the Day-1 benchmark. **$5** at store.ryanross.org.

What's inside

Thirty-three lessons, three checkpoints and a readiness final, listed in the order to work them. **Difficulty** is how hard the lesson is to get through. **Usefulness** is how often Calculus I will actually make you do it. The checkpoints and the final carry no rating because they are mixed and unlabeled on purpose: working out which tool a problem wants is half of what they test.

THE CALCULUS DOORWAY

CX1	**What a limit really is**	the name for a move you already own	★★☆☆☆	★★★★★
CX2	**Continuity**	when a graph has no surprises	★★☆☆☆	★★★★☆
CX3	**The derivative**	the thing you were already computing	★★★☆☆	★★★★★

LESSON 1 · DAY 1 · BENCHMARK

The Day-1 benchmark

This is not a lesson. It is a measurement, and you are **meant to fail it today**. Below are two problems that look impossible right now. That is the point: they are the two problems you will finish clean on **the last page of this workbook**. Today you attempt them and record how far you get. Every topic in this course is one of the tools those two problems need.

Here's the thing nobody tells you before calculus, and it's the whole reason this course exists: **people don't fail calculus because of the calculus.** They fail because their algebra and trig aren't automatic. In a typical problem, one step is the new idea and the other 90% is factoring, simplifying, and the unit circle. Both problems below are *pure Stage-1 language*. No limits. No derivatives. If you can finish them with no calculus, you are ready for Calculus I.

How to attempt these

> **Attempt each one honestly, cold, on paper.** No notes, no calculator, no internet. Push until you're genuinely stuck.

> **Being stuck is the measurement.** Where you stall *is* your baseline. It tells you exactly which topics to build first.

> **No full solutions here.** The reveals show you *which topics* each problem needs, not the answer. The worked solutions come in the readiness final at the end. Earning them is the course.

Benchmark 1: The Ugly Graph

Sketch this graph **using no calculus at all**. Just where it lives, where it dies, where it crosses zero, and where it heads far out:

$$\text{sketch} \quad f(x) = (x^2 - x - 6) / (x^2 - 4)$$

A real graph: the curve, any vertical walls it can't cross, any single point where it's secretly punched out, where it cuts the axes, and the flat line it hugs as x runs off to ±∞. Give it your best shot first.

which topics this problem needs (not the answer)

Every move here is a Stage-1 skill you're about to drill:

- **Factor the top and the bottom.** Both are quadratics in disguise. That's `TK1 · Factoring` . Once factored, something interesting cancels.
- **Find where the bottom hits zero.** That's the `FN3 · Domain` skill. It tells you the vertical wall *and* the one sneaky punched-out point (they are not the same thing, and knowing why is the whole game).
- **Read the sign on each side.** A `FN4 · sign-chart` tells you whether the curve dives up or down against each wall.
- **Look at the far-out behavior.** Compare the top and bottom's leading pieces. That's the flat line it approaches.

You'll have all four by the end of the Functions pillar. This exact problem is Checkpoint-1 territory.

Benchmark 2: The Equation That Looks Impossible

Find **every** value of x that solves this, not one answer but all of them:

$$2 \cdot \sin^2(x) + \sin(x) - 1 = 0$$

It has a squared trig term, a plain trig term, and a number. It looks like it belongs in a different universe than the graph above. It doesn't. It is the same two topics in a different costume.

which topics this problem needs (not the answer)

- **See the hidden quadratic.** Squint until `sin(x)` looks like a single variable. Then it's just `2u² + u − 1 = 0` , and that's `TK1 · Factoring` again. It splits into two simple pieces.
- **Read the unit circle.** Each piece leaves you asking "which angles have this sine?" That's `TR1 · Unit circle` , memorized cold.
- **Say "all of them."** Sine repeats forever, so each answer is really a whole family (the `+ 2πk` move), which is the periodicity idea from `TR2` .

Algebra you'll own in week one; the unit circle in the Trig pillar. This problem is where the two pillars meet.

Score your baseline

Be honest. This is just for calibration, and nobody is grading it. For each move, could you do it cleanly, cold, right now? Tap to see which topic builds it.

Factor $x^2 - x - 6$ without hesitating

→ `TK1 · Factoring I` (topic 1). If this was slow, it's the single highest-leverage thing to fix. Everything leans on it.

Know instantly why $x^2 - 4 = 0$ makes a *wall* at one x and a *hole* at another

→ `FN3 · Domain & where functions break`. The difference between a vertical asymptote and a hole is a favorite trap.

Build a sign-chart to tell if a curve goes up or down near a wall

→ `FN4 · Inequalities & sign-charts`. This is literally the ugly-graph method.

State sin(x) = 1/2 → all x, from memory

→ `TR1 · Unit circle + special angles`. Pure recall that pays off through every calculus class you'll take.

Rewrite √x as a power, or handle a negative exponent

→ `TK3 · Exponent & radical rules`. Not in either problem above, but the power rule in Calculus I dies without it, so it's in the Toolkit.

The sequence

Fourteen topics. You take roughly one every two days, drilling each to reflex before the next. Then you come back here.

Toolkit: factoring, factoring with ugly exponents, exponent/radical rules. *The alphabet.*

Functions: what a function is, the family of shapes, domain, sign-charts & the ugly-graph method, composition, exp/logs. *Benchmark 1 lives here.*

Trigonometry: the unit circle, the six functions as waves, the core identities. *Benchmark 2's second half lives here.*

> **Algebra in action**: rational expressions, solving equations & rationalizing. *The sentences you build from the alphabet.*

Three checkpoints along the way, then a mixed final that ends with **this exact ugly graph**, sketched clean. Beat it and you're done.

Now make your attempt. Work both benchmarks on paper and really try. Note how far you got on each (even "I factored the top and stalled" is gold). Then record your baseline:

Date attempted: __________ How far I got on Benchmark 1: __________ Benchmark 2: __________

Keep that note. You rematch these exact two problems in the readiness final at the end, and the distance between the two attempts is the entire point of this workbook. Next up: **TK1 · Factoring I**, the first topic. One at a time, each drilled to reflex before the next.

Answers

(And no, still no full solutions here either: the worked solutions come in the readiness final, by design.)

Benchmark 1 needs: factor top & bottom (TK1) → find where the bottom is zero, separating the wall from the hole (FN3) → sign-chart each side (FN4) → compare leading terms for the far-out flat line. Functions-pillar / Checkpoint-1 material.

Benchmark 2 needs: treat sin(x) as one variable and factor the quadratic (TK1) → read each piece off the unit circle (TR1) → write every solution as a repeating family (TR2). Where the Algebra and Trig pillars meet.

Baseline: the moves are built by TK1 (factoring), FN3 (domain: wall vs hole), FN4 (sign-charts), TR1 (unit circle), and TK3 (exponents/radicals). Slow on factoring? That's the top priority, because it's under everything.

Factoring I: un-multiplying

This is the first tool, and it is not an accident that it comes first. Look back at the Day-1 benchmark: *both* problems opened with a factoring move. The ugly graph needed the top and bottom factored before anything else made sense. The impossible-looking trig equation was a disguised $2u^2 + u - 1$. Factoring is the single most-used mechanical skill in Calculus I, and it is the one students most often arrive slow at.

The intuition

Multiplying is easy and factoring is hard, and that asymmetry is the whole story. $(x-3)(x+2)$ expands to $x^2 - x - 6$ in about four seconds. Going backwards (starting from $x^2 - x - 6$ and recovering the two pieces) is a small search problem, and searching is slower than following.

So why bother going backwards at all? Because of one fact that has no equal in algebra:

> **A product is zero exactly when one of its pieces is zero.** If $A \cdot B = 0$, then $A = 0$ or $B = 0$. There is no third option.

That rule only works on *products*. It says nothing about sums. $x^2 - x - 6 = 0$ is a sum of three things and gives up nothing. You can stare at it all day. But $(x-3)(x+2) = 0$ hands you the answers immediately: $x = 3$ and $x = -2$.

> *If you forget everything else in this lesson, remember this: factored form is the only form that gives up its roots without a fight. Everything you factor, you factor to get to that form.*

The second reason is canceling. In calculus you will constantly meet fractions that go to $0/0$, a shape that means nothing on its own. Factoring is how you find the common piece hiding on the top and bottom and kill it. That's TK2's job, next lesson, and it needs this one first.

The rigor: four moves, in order

Move 0: GCF first, always

Before anything clever, ask: is there a common factor in every term? Pull it out front. It costs five seconds and it routinely turns a hard problem into an easy one.

> $6x^3 - 9x^2$ → both terms carry a 3 and at least x^2 → $3x^2(2x - 3)$

Take the *lowest* power of x present (here x^2, not x^3), and the largest number dividing all coefficients. Always check your work by mentally re-multiplying.

Move 1: difference of squares

A pattern worth recognizing on sight, because it appears everywhere:

$$a^2 - b^2 = (a - b)(a + b)$$

> $x^2 - 4$ → x^2 and 4 are both perfect squares → $(x - 2)(x + 2)$
> $9x^2 - 25$ → squares of $3x$ and 5 → $(3x - 5)(3x + 5)$

Train the recognition, not the formula: *two terms, both perfect squares, separated by a minus sign.*

Move 2: simple trinomial (leading coefficient 1)

For $x^2 + bx + c$: find two numbers that **multiply to c** and **add to b**. Those two numbers are your constants.

> **Worked:** $x^2 - x - 6$, the benchmark's opener.
> Need two numbers with product -6, sum -1.
> Product is negative → the signs differ. Candidate pairs for 6: (1,6) and (2,3).
> Try -3 and $+2$: product -6 ✓, sum -1 ✓.
> → $(x - 3)(x + 2)$

Read the signs before you search. It halves the work. Product negative means one of each sign. Product positive means both match the sign of b.

Move 3: hard trinomial (leading coefficient ≠ 1), the AC method

For $ax^2 + bx + c$ the guess-and-check gets ugly fast. Use this instead, and it never fails:

> **Worked:** $6x^2 + x - 12$
> **1.** Multiply $a \cdot c$ = $6 \cdot (-12)$ = -72.
> **2.** Find two numbers with product -72, sum $+1$ (the b): that's $+9$ and -8.
> **3.** Split the middle term using them: $6x^2 + 9x - 8x - 12$.
> **4.** Factor by grouping, two terms at a time: $3x(2x + 3) - 4(2x + 3)$.
> **5.** The bracket now repeats. Pull it out: $(3x - 4)(2x + 3)$.

Step 5 is the payoff and also the self-check: **if the two brackets don't come out identical, you made an arithmetic slip earlier.** The method tells you when it has gone wrong, which is why it beats guessing.

Where this goes wrong

> **A sum of squares does not factor.** $x^2 + 4$ is prime over the real numbers. There is no $(x + 2)(x + 2)$. Check it: that expands to $x^2 + 4x + 4$. Only the *difference* of squares splits.

> **Skipping the GCF.** $2x^3 - 8x$ looks like nothing until you pull the $2x$: $2x(x^2 - 4)$, and now a difference of squares is sitting there in plain sight. Students who skip Move 0 routinely declare things unfactorable.

> **Stopping one step early.** Factoring means factoring *completely*. $2x(x^2 - 4)$ is not the final answer. $x^2 - 4$ still splits. After every factorization, look at each piece and ask "can this one go further?"

> **Sign drift in the AC method.** When you split the middle and the third term is negative, the grouping needs a minus factored out: $- 8x - 12$ becomes $-4(2x + 3)$, not $-4(2x - 3)$. Factoring out a negative flips the sign of everything inside it.

Now drill it: 5 problems

1. Factor completely: $12x^4 - 18x^3$

WORK

2. Factor: $x^2 - 5x - 14$

WORK

3. Factor: $49x^2 - 16$

WORK

4. Factor: $6x^2 + x - 12$ (AC method, show the split)

WORK

5. Factor *completely*: $2x^3 - 8x$

WORK

Score yourself. Do all five cold, on paper, no notes. Then check against the Answers page and count how many you got *fully* right. A partially-factored answer counts as a miss, because in calculus it is one.

Score on paper: _______ / 5

That marks **TK1 complete** and opens **TK2: Factoring II**, where these same moves get pointed at the ugly exponents calculus actually throws.

Primary source

For extra reps *after* this lesson: **Khan Academy Algebra I, "Quadratics: Multiplying & factoring"** and the follow-on "Factoring quadratics" unit. Free, and the practice sets are auto-generated so you can drill Move 2 and Move 3 to fluency. Use it for volume; the understanding is meant to happen here.

Answers

1. $6x^3(2x - 3)$. GCF 6 divides both coefficients, x^3 is the lowest power.
2. $(x - 7)(x + 2)$. Two numbers with product -14, sum -5 → $-7, +2$.
3. $(7x - 4)(7x + 4)$. Difference of squares, $a = 7x$, $b = 4$.
4. $(3x - 4)(2x + 3)$. $a \cdot c = -72$; $+9$ and -8; split to $6x^2 + 9x - 8x - 12$; group to $3x(2x + 3) - 4(2x + 3)$. Verify by expanding: $6x^2 + 9x - 8x - 12 = 6x^2 + x - 12$ ✓
5. $2x(x - 2)(x + 2)$. GCF $2x$ first, then $x^2 - 4$ is a difference of squares. Stopping at $2x(x^2 - 4)$ is incomplete.

LESSON 3 · FN1 · FUNCTIONS

What a function really is, and how to read one off a graph

The Toolkit is complete. Now the star of the show.

DIBEOS calls reading a graph the highest-leverage skill in the whole roadmap, and here's why that claim holds. Calculus is not really about limits, derivatives, and integrals. Those are the tools. Calculus is about *functions*, and every one of those tools is a question you ask a function. If the function itself is fuzzy to you, every answer will be fuzzy too, no matter how well you execute the mechanics.

This is also the lesson where the most damage gets undone. Most students carry a definition of "function" that is really a definition of "formula," and that mismatch quietly breaks things for two semesters.

The intuition

A function is a **machine with one unbreakable promise**: feed it one input, and it gives back exactly one output. Same input tomorrow, same output tomorrow. No moods.

That's it. That is the entire definition. Notice what it does *not* say. It says nothing about formulas. A function doesn't need to be `x² + 3`. It can be a table, a graph drawn by hand, a rule in words ("the temperature at time t"), or a physical process. The formula is one costume a function can wear, not the function itself. Students who silently believe "function = formula" get stuck the first time a problem hands them a graph and no equation.

The promise (one input, one output) is the whole content, and it's what the **vertical line test** checks. Drop a vertical line anywhere on a graph. That line is you asking "at this input, what comes out?" If the line hits the curve twice, the graph just gave two answers to one question, and it broke the promise. Not a function. The test isn't a rule to memorize; it's the definition, drawn.

The shadows

Here's the picture worth keeping. Imagine a light shining straight down on the curve: the shadow it casts on the horizontal axis is the **domain**, every input the function actually accepts. Now shine the light from the side: the shadow on the vertical axis is the **range**, every output it actually produces.

> **Domain** = the curve's shadow on the x-axis (what you're allowed to put in).
> **Range** = the curve's shadow on the y-axis (what actually comes out).

Getting these from a picture instead of from algebra is the fastest way to make them stop feeling like vocabulary.

> *If you forget everything else: a function is a promise of one output per input, and a graph is that promise drawn, so every question about a function is a question you can ask the picture.*

The rigor: notation and what to read

Function notation

`f(x)` is read "f of x" and means *the output of machine f when the input is x*. It is **not** f multiplied by x. That single misreading causes an enormous amount of wreckage.

> `f(3) = 7` means: put 3 in, get 7 out. On the graph, the point `(3, 7)` sits on the curve.
> `f(a + h)` means: put the whole quantity `a + h` into the machine. Wherever the rule says x, write `(a + h)`, brackets and all.

That second line is not idle. `f(a+h)` is literally the first thing inside the definition of a derivative, and students who fumble it in precalculus fumble the derivative definition three months later.

What a graph will be asked to tell you

> **Value:** `f(2)`. Go to `x = 2`, go up or down to the curve, read the height.
> **Zeros / roots:** where the curve *crosses the x-axis*, the inputs making `f(x) = 0`.
> **y-intercept:** the height at `x = 0`, i.e. `f(0)`. There is at most one (the promise!).
> **Increasing / decreasing:** read *left to right*. Uphill = increasing, downhill = decreasing.
> **Sign:** where the curve sits *above* the axis, `f(x) > 0`; below it, `f(x) < 0`.

Those five readings are the vocabulary of every graph question you'll be asked, in this course and in Calc I. Note how much you can say about a function with no formula anywhere in sight.

Where this goes wrong

> **`f(x + 2)` is not `f(x) + 2`.** The first changes what goes *into* the machine; the second changes what comes *out*. On a graph, one shifts the curve sideways and the other shifts it up. FN2 makes this precise. For now, just never treat them as the same thing.

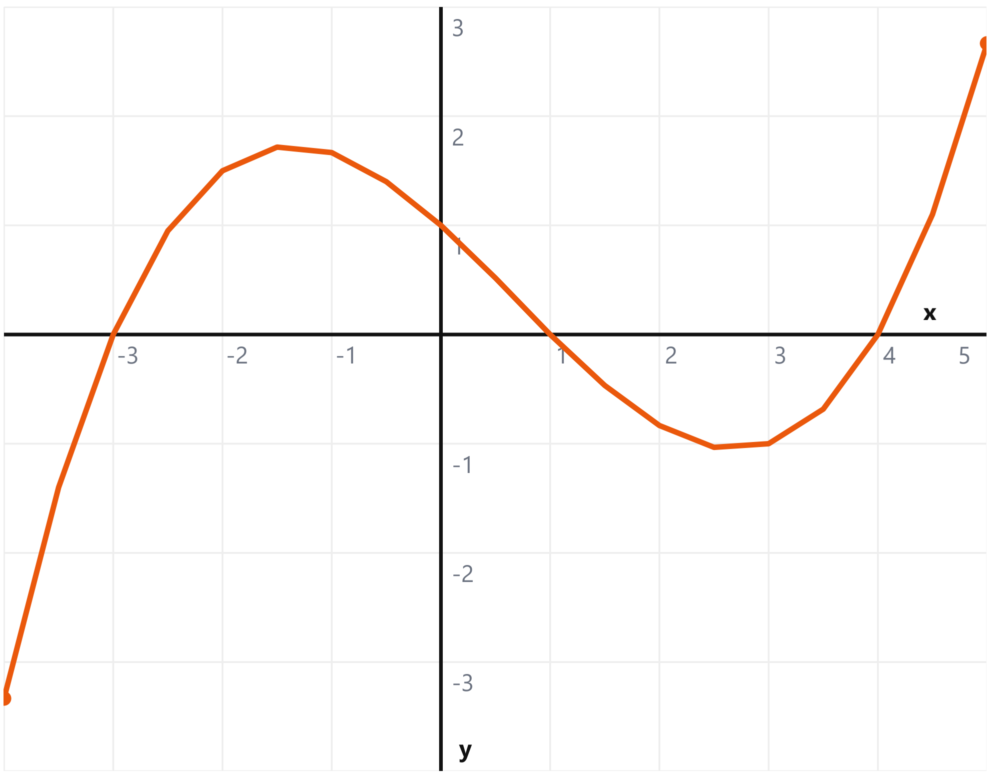
> **Confusing the two intercepts.** x-intercepts are where `f(x) = 0`. You're solving for the input. The y-intercept is `f(0)`. You're evaluating at zero. A function can have many x-intercepts but only one y-intercept, because two would break the promise.

> **Reading increase/decrease right to left.** Always sweep left to right, the direction x grows. A curve falling as you read leftward is *increasing*.

> **Assuming the domain is "all real numbers" by reflex.** On a drawn graph the domain is only what's actually drawn. From a formula it's whatever doesn't break, and FN3 is entirely about the two ways things break.

Now explain it back: 5 items

Items 1–3 use the graph below. Items 4–5 are pure explain-back: say the answer *out loud in full sentences*, then write the short version on the lines and check it.



The graph of `y = f(x)`. The solid dots at each end mark where the graph starts and stops.

1. State the **domain** of `f` from the graph, and give `f(0)`.

ANSWER

2. List all **zeros** of `f` (the x-intercepts).

ANSWER

3. On which intervals is $f(x) < 0$? (Read where the curve sits *below* the axis.)

ANSWER

4. Explain, without using the word "formula," **what makes something a function**, and why the vertical line test detects it.

ANSWER

5. If $f(x) = x^2 - 1$, write out $f(a + h)$ in full, and state in one sentence why $f(x + 2)$ and $f(x) + 2$ are different.

ANSWER

Score yourself. This one is about explaining rather than computing, so the bar is different from the Toolkit drills: close the page and say all five back *out loud, in full sentences, with no notes*. Reading an answer and nodding is not the same as producing it, and only one of those is what an exam asks for. Stumble somewhere? Reopen, find it, close, try again.

Score on paper: _______ / 5

That marks **FN1 complete** and opens **FN2: The function library + transformations**, where you collect the parent shapes and learn to move them. The Toolkit is finished; the Functions pillar begins.

Primary source

For extra reps *after* this lesson: **Khan Academy Algebra I, "Functions"** (the "Evaluating functions," "Inputs and outputs of a function," and "Interpreting function graphs" sets). The graph-interpretation exercises in particular give you far more reps at reading a picture than any textbook section will. Use it for volume; the understanding is meant to happen here.

Answers

1. **Domain** `[-4, 5]`; the solid endpoint dots mean both ends are included. `f(0) = 1`.
2. `x = -3`, `x = 1`, `x = 4`.
3. `[-4, -3)` **and** `(1, 4)`. Below the axis at the far left before the first crossing, and again between the second and third crossings.
4. **Function: a rule giving each input exactly one output.** The vertical line test asks "what comes out at this input?". A line meeting the curve twice means two outputs for one input, which breaks the definition. The test *is* the definition, drawn.
5. `f(a + h) = (a + h)² − 1 = a² + 2ah + h² − 1`. `f(x + 2)` alters the input before the machine runs (horizontal shift); `f(x) + 2` alters the output after (vertical shift).

LESSON 4 · PF6 · TRIG

Sine and cosine: the two waves, rebuilt from the key points

Trig graphs are where a lot of people lose marks, and it's worth being precise about why. It's almost never that sine and cosine are beyond them. It's that they walked in cold, having memorized a picture instead of learning to rebuild one. That's the good news, because nothing here is hard once you stop trying to *memorize the picture*. So we're rebuilding both graphs from scratch, from five points each, in a way that survives a cold exam.

Here's the reframe that fixes trig graphs for good. You never memorize what a sine wave looks like and hope you can redraw the wiggle. You plot **five points** across one lap and connect them with a smooth curve. Five points, in order. That is the entire skill, and it's the same five-point move for cosine.

The intuition

Sine and cosine are the **same wave**. Genuinely the same shape. One just starts in a different spot than the other. Both of them roll left and right forever, both stay trapped between `−1` and `1`, and both repeat exactly every `2π` (one full trip around a circle). Nothing about either one ever escapes that strip or breaks that rhythm.

Because they're bounded and repeating, three numbers describe either graph completely, and you can read all three straight off the five points:

> **Midline** = the horizontal center line the wave rocks around. For the parent graphs it's `y = 0`.
> **Amplitude** = how far up (and down) from the midline the wave reaches. For the parents it's `1`.
> **Period** = how wide one full repeat is before the pattern starts over. For the parents it's `2π`.

Plot five points, connect a smooth wave, and those three readings are just *there* in the picture. No memorized shape required.

> *Sine and cosine are bounded waves: they never leave [−1, 1] and they repeat every 2π. Learn the five key points of one period and you can draw either one, and read amplitude, period, and midline straight off them.*

The rigor: the five points, then how to move them

Three facts. After these you can draw either graph and read every number off it.

> **`y = sin x`**: Domain all reals; Range `[−1, 1]`; period `2π`; midline `y = 0`; amplitude `1`.
> The five key points over one period: `(0, 0)` → `(π/2, 1)` → `(π, 0)` → `(3π/2, −1)` → `(2π, 0)`.
> **It starts at zero and climbs.** Up to the top at `π/2`, back to the middle at `π`, down to the bottom at `3π/2`, home at `2π`.

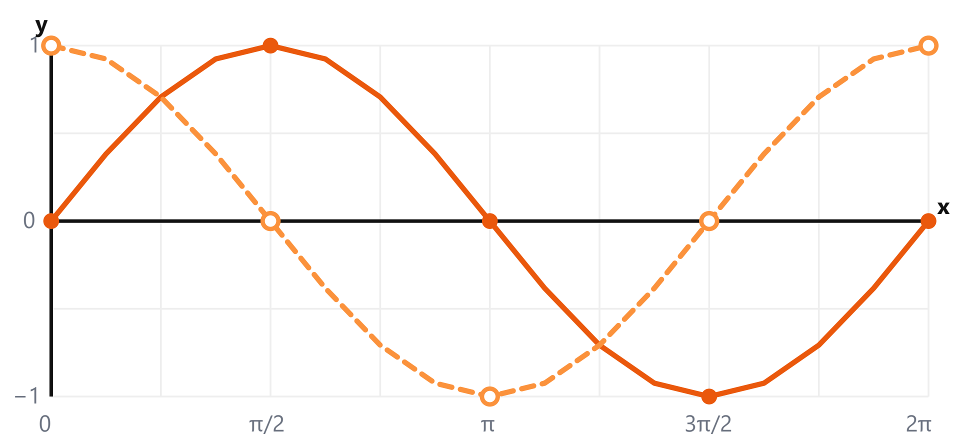
`y = cos x` : Domain all reals; Range `[−1, 1]` ; period `2π` ; midline `y = 0` ; amplitude `1` .
The five key points: `(0, 1)` → `(π/2, 0)` → `(π, −1)` → `(3π/2, 0)` → `(2π, 1)` .
It starts at its maximum. Cosine is just sine slid left by `π/2` , same wave, launched from the top instead of the middle.

Everything else is one formula. Take either parent and dress it up as `y = A·sin(Bx) + D` (or the same with cosine); each knob does exactly one job:

Transformations, `y = A·sin(Bx) + D` (identical for cosine):
Amplitude = |A|, stretches the *height*. The wave now reaches `|A|` above and below the midline.
Period = 2π/B, squeezes the *width*. A bigger `B` packs the repeats closer together.
Midline = y = D, lifts (or lowers) the whole wave by `D` .
Range = [D − |A|, D + |A|], the midline, plus and minus the amplitude.

The two waves, one period

Here is exactly what those key points draw. Trace it: sine leaves the origin climbing; cosine starts pinned at the top. The dots are the five key points you actually plot. Everything between them is just a smooth connect-the-dots.



One period, `0` to `2π` . **Solid orange =** `y = sin x` , starts at 0 and climbs; its dots are the filled circles. **Dashed =** `y = cos x` , starts at its maximum, 1; its dots are the open circles. Both stay inside the strip `[−1, 1]` and both repeat every `2π` . Notice cosine is just sine slid left by `π/2` .

Where this goes wrong

> **The range is `[-1, 1]`, not all reals.** The *domain* is all reals: you can feed in any angle. But the outputs are trapped: these are bounded waves, and they cannot escape the strip between `-1` and `1`. Writing the range as "all reals" is the reflex that costs the most points.

> **The parent period is `2π`, not `π`.** Sine does return to zero at `π`, but that's only halfway through the lap. A *full repeat*, where the whole pattern starts over identically, takes `2π`. Don't let the mid-cycle zero fool you into halving the period.

> **Amplitude and period are not the same knob.** Amplitude (the `A`) scales the *height*, how far up and down. Period (the `2π/B`) scales the *width*, how far sideways before it repeats. One stretches the range, the other stretches the wavelength. Swapping them is a classic trig-graph slip.

> **Cosine starts at its maximum `(0, 1)`; sine starts at `(0, 0)` and climbs.** Mixing up the starting point is the single most common parent-graph mistake. When in doubt: cosine is already at the top when `x = 0`, sine is dead-center and heading up.

Now drill it: 5 problems

1. State the **domain** and **range** of `y = sin x`.

WORK

2. For `y = 3 sin x`, give the **amplitude** and the **range**.

WORK

3. For `y = cos(2x)`, give the **period**.

WORK

4. For `y = sin x + 2`, give the **midline** and the **range**.

WORK

5. On the interval $[0, 2\pi]$, where does $\cos x = 0$?

WORK

Score yourself. Do all five cold, on paper, no notes. Then check against the Answers page and count how many you got *fully* right. Writing the range as "all reals" or the parent period as π counts as a miss, because killing those two slips is the whole point of this unit.

Score on paper: _______ / 5

That marks **PF6 complete** and adds sine & cosine to your **parent-functions strand**, the shelf of standard shapes you can draw on command. Next time a wave shows up in a problem, you don't panic; you plot five points.

Primary source

For extra reps *after* this lesson: **Khan Academy "Graphs of sin, cos & tan."** Once the five-point method feels automatic here, go there for volume: the practice sets throw amplitudes, periods, and midlines at you until reading them off is reflex. Use it for reps; the understanding is meant to happen on this page.

Answers

1. **Domain** $(-\infty, \infty)$ **; Range** $[-1, 1]$ **.** Any angle goes in; outputs stay trapped between -1 and 1 because sine is a bounded wave.
2. **Amplitude** 3 **; Range** $[-3, 3]$ **.** The $A = 3$ stretches the height to 3 above and below the midline $y = 0$.
3. **Period** $= 2\pi/2 = \pi$ **.** Here $B = 2$, and period is $2\pi/B$; the inside 2 squeezes one full lap into half the width.
4. **Midline** $y = 2$ **; Range** $[1, 3]$ **.** The $D = +2$ lifts the whole wave up by 2; amplitude is still 1, so range is $[2 - 1, 2 + 1]$.
5. $x = \pi/2$ **and** $x = 3\pi/2$ **.** Those are cosine's two midline crossings on $[0, 2\pi]$. The key points $(\pi/2, 0)$ and $(3\pi/2, 0)$.

What a limit really is: the name for a move you already own

You have already been taking limits without knowing the word for it. Every difference quotient you canceled back in `TK9` (combining the stacked fraction, killing the `h`, then seeing what was left) was a limit. This is the doorway lesson: no new algebra, no grind. Its whole job is to hand you the *name and the picture* for a move you've already got in your hands, because that name is the first word of calculus and it turns out you already speak it.

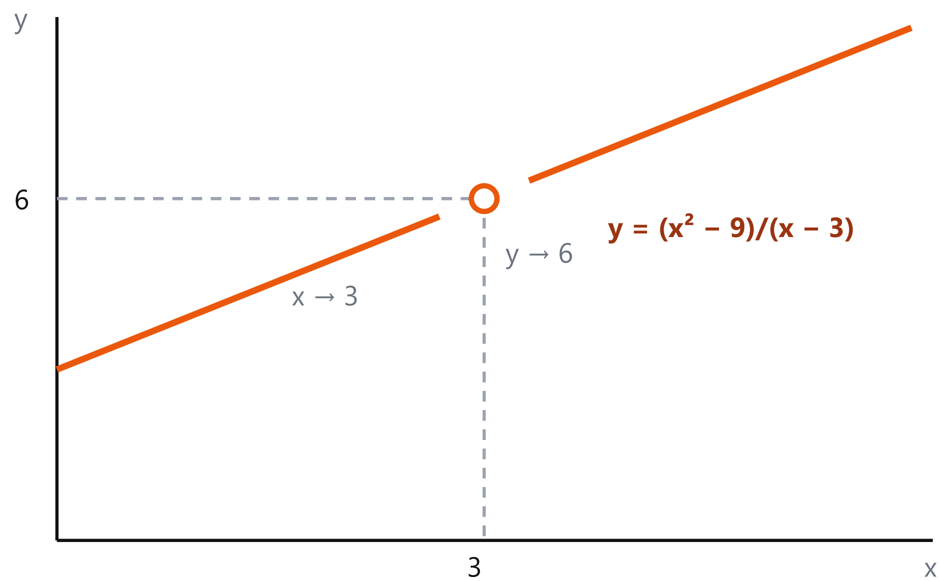
The intuition

A **limit** asks one question: *as the input gets closer and closer to some value, what is the output heading toward?* Notice what it does **not** ask: it does not ask what the function *is* at that point. It asks where the function is *going* near that point. Those are different questions, and the gap between them is the entire reason limits exist.

Why would you ever care where a function is "heading" instead of just plugging the point in? Because at the points that matter most in calculus, **plugging in gives you `0/0`**, nothing. That's not a dead end; it's a doorbell. It means: *stop, do some algebra, and ask where the thing was heading instead*. You have already met this exact wall:

> **You've already lived this.** The difference quotient `(f(x+h) − f(x)) / h` at `h = 0` is `0/0`, meaningless. But you didn't stop there. You canceled the `h`, and the leftover expression pointed at a single clean number. That number is *the limit as* `h → 0`, and you now know it's called the derivative. You were computing limits the whole time.

Here is the picture that makes it concrete. Take `f(x) = (x² − 9)/(x − 3)`. At `x = 3` it's `0/0`, genuinely undefined, a hole in the graph. But everywhere else it's just the line `y = x + 3` (factor the top: `(x−3)(x+3)/(x−3)`, cancel). So as `x` creeps toward `3` from either side, the output creeps toward `3 + 3 = 6`, even though the function never actually *reaches* `x = 3`:



The open circle at `(3, 6)` is the hole: `f(3)` is `0/0`, undefined. But from both sides the curve marches straight at height `6`. The function never lands on the point, yet it is unmistakably *heading* there. That heading-toward value, `6`, is the limit.

> *A limit is the value a function heads toward as its input approaches a point, not the value at the point. When plugging in gives 0/0, the limit is the real answer hiding behind it, and algebra is how you uncover it.*

The rigor: the notation and the three cases

We write it like this:

> $\lim_{x\to a} f(x) = L$ reads "as `x` approaches `a`, `f(x)` approaches `L`."
> The arrow `x → a` means `x` gets *arbitrarily close to* `a`, but never has to equal it. That "never has to equal it" is the whole point: the limit ignores what happens *at* `a` and watches only the approach.

When you actually evaluate one, you land in exactly one of three cases. Recognizing which is the entire skill:

> **Case 1: plug in and you get a clean number.** If the function is well-behaved at `a` (no division by zero, no even root of a negative), the limit is just `f(a)`. Example: $\lim_{x\to 2} (x^2 + 1) = 5$. Most limits are this boring, and boring is good.

> **Case 2: plug in and you get `0/0`.** This is **indeterminate**: not zero, not undefined-forever, just "not done yet." It is a signal to do algebra (factor and cancel, or run the difference-quotient move) until the trouble cancels, *then* plug in. Example: $\lim_{x\to 3} (x^2-9)/(x-3)$ → factor → $\lim_{x\to 3} (x+3) = 6$. This is the case calculus is built on, and it's the one you drilled in `TK9`.

> **Case 3: plug in and you get (nonzero)/0.** Like `1/0`. This is *not* indeterminate: it blows up to `±∞`, a **vertical asymptote**. The two-sided limit does not exist as a number. Example: $\lim_{x\to 2} 1/(x-2)$ shoots to `+∞` from the right and `−∞` from the left.

One-sided limits: the left and the right

You can approach `a` from below ($x \to a^-$, from the left) or from above ($x \to a^+$, from the right). The full two-sided limit exists **only if both sides agree** on the same number. In the hole picture above, both sides marched to `6`, so the limit is `6`. In Case 3, the left went to `−∞` and the right to `+∞`. They disagree, so there's no two-sided limit.

This is the hole-vs-wall distinction you already own

Back in `FN3` you separated a *hole* from a *vertical wall* and nailed it cold on the benchmark. Limits are just the precise language for that instinct: a **hole** is a `0/0` that cancels, the limit exists (a real number) but `f(a)` doesn't. A **wall** is a `(nonzero)/0`: the limit is infinite. Same two shapes you already read off a graph, now with names. You're not learning a new idea here; you're getting the vocabulary for three things you already do.

Where this goes wrong

> **Thinking the limit is just `f(a)`.** It usually equals `f(a)`, but only in Case 1, and calculus lives in Case 2 where `f(a)` doesn't even exist. The limit is about the *approach*, not the point. Conflating the two is the mistake that makes limits feel pointless; keeping them separate is the whole idea.

> **Stopping at `0/0` and writing "undefined."** `0/0` is indeterminate, a doorbell, not an answer. It means *do more work*. Every time you see it, your hand should reach for factoring or the difference-quotient move, exactly as it did in the Toolkit lessons. The answer is on the other side of the algebra.

> **Ignoring a one-sided disagreement.** If the left approach and the right approach head to different values, there is *no* two-sided limit. You can't average them or pick one. Always check that both sides agree before you claim a limit exists.

Now drill it: 5 problems

1. Evaluate: $\lim_{x\to 2} (x^2 + 1)$

WORK

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2. Evaluate: $\lim_{x \to 3} (x^2 - 9)/(x - 3)$

WORK

3. Evaluate: $\lim_{x \to 1} (x - 1)/(x^2 - 1)$

WORK

4. Evaluate: $\lim_{h \to 0} ((3 + h)^2 - 9)/h$

WORK

5. What is $\lim_{x \to 2} 1/(x - 2)$, and what does it tell you about the graph?

WORK

Score yourself. These are a concept check, not a grind: the win is naming the case (1, 2, or 3) correctly and finishing it. Do all five on paper, then check against the Answers page and count how many you got fully right, case-name included.

Score on paper: _______ / 5

That marks **CX1 complete**. You've stepped through the doorway. You now have the word for what you were already doing: **limit**. Next is `CX2` (continuity, when a function has no holes or walls), then `CX3`, where the difference quotient officially becomes **the derivative**.

Primary source

For the picture done as motion: **3Blue1Brown "The paradox of the derivative"** (Essence of Calculus, ch. 1–2) shows the limit as a value approached, in animation, better than any static page can. And **Khan Academy: "Limits**

intro" for extra worked reps across the three cases. Watch/use them after: the understanding is meant to land here.

Answers

1. **`5`.** Case 1. Nothing breaks at `x = 2`; plug in: `2² + 1 = 5`.
2. **`6`.** Case 2 (`0/0`). Factor: `(x−3)(x+3)/(x−3) → x + 3`; plug in `3` → `6`. The canceled `(x−3)` is the hole.
3. **`1/2`.** Case 2 (`0/0`). Factor the bottom: `(x−1)/[(x−1)(x+1)] → 1/(x+1)`; plug in `1` → `1/2`.
4. **`6`.** Case 2 (`0/0`), a difference quotient. `((3+h)²−9)/h = (6h + h²)/h = 6 + h`; let `h → 0` → `6`. This is the derivative of `x²` at `x = 3`.
5. **Does not exist** (two-sided); **vertical asymptote at `x = 2`**. Case 3 (`1/0`): `+∞` from the right, `−∞` from the left. The sides disagree, so no two-sided limit. It's the FN3 wall in limit language.

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